Spherical Wave Approach (SWA) for the electromagnetic modelling of 3D GPR scenarios

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Outline

- Solutions of the vector wave equation
- Spherical wave approach
- Spherical reflected and transmitted waves
- Results
- Conclusions

Vector wave equation

The wave equation in free space (eventually lossy!)

$$\nabla^2 \underline{C} - \mu\epsilon \frac{\partial^2 \underline{C}}{\partial t^2} - \mu\sigma \frac{\partial \underline{C}}{\partial t} = 0$$

Where $\underline{C}$ represents a generic electromagnetic field.

In the frequency domain, in the most general case:

$$\nabla\nabla \cdot \underline{C} - \nabla \times \nabla \times \underline{C} + k^2 \underline{C} = 0$$

Unfortunately, the solution of such equation is not easy at all!

Scalar wave equation

On the other hand, the scalar equation:

$$\nabla^2\psi + k^2\psi = 0$$

Is very easy to solve, at least in “simple” coordinate systems

Examples:

- Cartesian $Ae^{j\underline{k}\cdot\underline{r}} + Be^{-j\underline{k}\cdot\underline{r}}$
- Cylindrical $\left[AJ_m(k_\rho\rho) + BY_m(k_\rho\rho)\right]e^{-jm\phi}e^{-jk_z}$
- Spherical $[Aj_n(kr) + By_n(kr)]e^{-jm\phi}P_n^m(\cos\theta)$

Vector wave equation

Exists a really clever way to obtain the solution of the vector wave equation, starting from the scalar one!

Let us define the following vectors:

$$\underline{L} = \nabla \psi \qquad \underline{M} = \nabla \times (\underline{a}_0 \psi) \qquad \underline{N} = \frac{1}{k} \nabla \times \underline{M}$$

Where ψ represents a solution of the scalar wave equation, and $\underline{a}_0$ is a constant vector.

Each of these vectors is a solution of the vector wave equation!

Precious properties

$$\underline{L} = \nabla \psi \qquad \underline{M} = \nabla \times (\underline{a}_0 \psi) \qquad \underline{N} = \frac{1}{k} \nabla \times \underline{M}$$

These vector functions have very important properties:

$$\underline{M} = \frac{1}{k} \nabla \times \underline{N} \qquad \nabla \cdot \underline{L} = \nabla^2 \psi = -k^2 \psi$$

$$\nabla \times \underline{L} = 0 \qquad \nabla \cdot \underline{M} = 0 \qquad \nabla \cdot \underline{N} = 0$$

Series expansion

As it is well known, a set of solutions of the scalar wave equation (eigenvectors) forms a base of the L^2 space.

At this set, it is associated a set of vector, and any vector function would be written as a superposition of such vectors.

If, for example, we consider the magnetic vector potential $\underline{A}$:

$$\underline{A} = \frac{1}{i\omega\mu} \sum_{n=0}^{+\infty} (a_n \underline{M}_n + b_n \underline{N}_n + c_n \underline{L}_n)$$

Field expansion

$$\underline{A} = \frac{1}{i\omega\mu}\sum_{n=0}^{+\infty}\left(a_n\underline{M}_n + b_n\underline{N}_n + c_n\underline{L}_n\right)$$

If now we consider the electric and magnetic fields:

$$\underline{H} = \nabla \times \underline{A} \qquad \underline{E} = -\frac{1}{i\omega\epsilon}\nabla \times \underline{H}$$

Hence:

$$\underline{E} = \sum_{n=0}^{+\infty}\left(a_n\underline{M}_n + b_n\underline{N}_n\right)$$

$$\underline{H} = \frac{k}{i\omega\mu}\sum_{n=0}^{+\infty}\left(a_n\underline{N}_n + b_n\underline{M}_n\right)$$

Field expansion

$$\underline{A} = \frac{1}{i\omega\mu}\sum_{n=0}^{+\infty}\left(a_n\underline{M}_n + b_n\underline{N}_n + c_n\underline{L}_n\right)$$

If now we consider the electric and magnetic fields:

$$\underline{H} = \nabla \times \underline{A} \qquad \underline{E} = -\frac{1}{i\omega\epsilon}\nabla \times \underline{H}$$

Hence:

$$\boxed{\nabla \times \underline{L} = 0}$$

$$\underline{E} = \sum_{n=0}^{+\infty}\left(a_n\underline{M}_n + b_n\underline{N}_n\right)$$

$$\underline{H} = \frac{k}{i\omega\mu}\sum_{n=0}^{+\infty}\left(a_n\underline{N}_n + b_n\underline{M}_n\right)$$

Field expansion

$$\underline{A} = \frac{1}{i\omega\mu} \sum_{n=0}^{+\infty} (a_n \underline{M}_n + b_n \underline{N}_n + c_n \underline{L}_n)$$

If now we consider the electric and magnetic fields:

$$\underline{H} = \nabla \times \underline{A} \qquad \underline{E} = -\frac{1}{i\omega\epsilon} \nabla \times \underline{H}$$

$$\boxed{\underline{M} = \frac{1}{k} \nabla \times \underline{N}}$$

$$\boxed{\underline{N} = \frac{1}{k} \nabla \times \underline{M}}$$

Hence:

$$\underline{E} = \sum_{n=0}^{+\infty} (a_n \underline{M}_n + b_n \underline{N}_n)$$

$$\underline{H} = \frac{k}{i\omega\mu} \sum_{n=0}^{+\infty} (a_n \underline{N}_n + b_n \underline{M}_n)$$

Complex but simple

$$\underline{E} = \sum_{n=0}^{+\infty} (a_n \underline{M}_n + b_n \underline{N}_n)$$

$$\underline{H} = \frac{k}{i\omega\mu} \sum_{n=0}^{+\infty} (a_n \underline{N}_n + b_n \underline{M}_n)$$

We must recall that the functions $\underline{M}$ and $\underline{N}$ are known by the simple solution of the scalar wave equation. The only unknowns are the expansions' coefficients!

Complex but simple

$$\underline{E} = \sum_{n=0}^{+\infty} (a_n \underline{M}_n + b_n \underline{N}_n)$$

$$\underline{H} = \frac{k}{i\omega\mu} \sum_{n=0}^{+\infty} (a_n \underline{N}_n + b_n \underline{M}_n)$$

“Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture…”

B. Russel

Plane waves

$$\psi(\underline{r}) = e^{i\underline{k}\cdot\underline{r}}$$

We can easily obtain the vector functions:

$$\underline{L} = i\psi\underline{k} \qquad \underline{M} = i\psi\underline{k} \times \underline{a}_0 \qquad \underline{N} = \frac{1}{k}\psi(\underline{k} \times \underline{a}_0) \times \underline{k}$$

But we can consider plane-wave spectrum:

$$\psi(\underline{r}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(k_x, k_y)e^{i\underline{k}\cdot\underline{r}}dk_x dk_y = \int\int g(\alpha, \beta)e^{i\underline{k}\cdot\underline{r}}d\alpha d\beta$$

Plane-wave spectrum

Inserting the vector functions expressions in the integral:

$$\underline{L} = i \int\int g(\alpha, \beta)\underline{k}(\alpha, \beta)e^{i\underline{k}\cdot\underline{r}}d\beta d\alpha$$

$$\underline{M} = i \int\int g(\alpha, \beta)\underline{k}(\alpha, \beta) \times \underline{a}_0 e^{i\underline{k}\cdot\underline{r}}d\beta d\alpha$$

$$\underline{N} = \frac{1}{k} \int\int g(\alpha, \beta)[\underline{k}(\alpha, \beta) \times \underline{a}_0] \times \underline{k}(\alpha, \beta)e^{i\underline{k}\cdot\underline{r}}d\beta d\alpha$$

We have, for free, the plane-wave spectra of such functions!

Spherical coordinates

The solution of the scalar wave equation in spherical coordinates can be written as follows:

$$\psi_{mn}(\underline{r}) = D z_n(kr) P_n^m(\cos\theta) e^{im\phi}$$

Where z_n represents a generic spherical Bessel function, hence:

$$\underline{M}_{mn} = \frac{im}{\sin\theta} z_n(kr) P_n^m(\cos\theta) e^{im\phi} \underline{\theta}_0 - z_n(kr) \frac{\partial P_n^m(\cos\theta)}{\partial\theta} e^{im\phi} \underline{\phi}_0$$

$$\underline{N}_{mn} = \frac{z_n(kr)}{kr} n(n+1) P_n^m(\cos\theta) e^{im\phi} \underline{r}_0 + \frac{1}{kr} \frac{\partial[r z_n(kr)]}{\partial r} \frac{\partial P_n^m}{\partial\theta} e^{im\phi} \underline{\theta}_0 + \frac{1}{kr} \frac{\partial[r z_n(kr)]}{\partial r} \frac{im}{\sin\theta} P_n^m e^{im\phi} \underline{\phi}_0$$

Auxiliary functions

Let us introduce the following functions:

$$\pi_{mn}(\theta) = m\frac{P_n^m(\cos\theta)}{\sin\theta}$$

$$\tau_{mn}(\theta) = \frac{dP_n^m(\cos\theta)}{d\theta}$$

Then, we can define the so-called vector tesseral functions:

$$\underline{m}_{mn}(\theta,\phi) = e^{im\phi}\left[i\pi_{mn}(\theta)\underline{\theta}_0 - \tau_{mn}(\theta)\underline{\phi}_0\right]$$

$$\underline{p}_{mn}(\theta,\phi) = e^{im\phi}n(n+1)P_n^m(\cos\theta)\underline{r}_0$$

$$\underline{n}_{mn}(\theta,\phi) = e^{im\phi}\left[\tau_{mn}(\theta)\underline{\theta}_0 + i\pi_{mn}(\theta)\underline{\phi}_0\right]$$

Precious properties

The vector tesseral functions are orthogonal:

$$\int_0^{2\pi}\int_0^{\pi} \underline{m}_{mn} \cdot \underline{n}^*_{m'n'} \sin\theta d\theta d\phi = 0$$

$$\int_0^{2\pi}\int_0^{\pi} \underline{m}_{mn} \cdot \underline{m}^*_{m'n'} \sin\theta d\theta d\phi = 4\pi \frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{mm'}\delta_{nn'}$$

$$\int_0^{2\pi}\int_0^{\pi} \underline{n}_{mn} \cdot \underline{n}^*_{m'n'} \sin\theta d\theta d\phi = 4\pi \frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{mm'}\delta_{nn'}$$

“Simpler” expressions

Thanks to the vector tesseral functions:

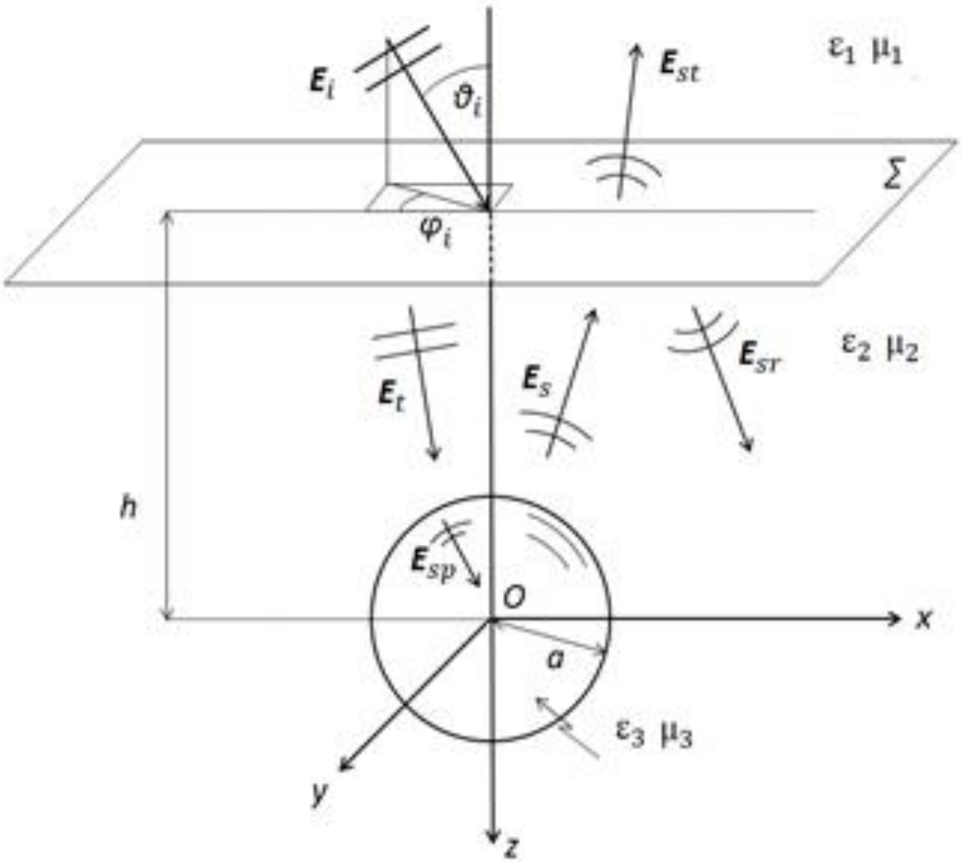
$$\underline{M}_{mn}(r,\theta,\phi) = z_n(\rho)\underline{m}_{mn}(\theta,\phi)$$

$$\underline{N}_{mn}(r,\theta,\phi) = \frac{z_n(\rho)}{\rho}\underline{p}_{mn}(\theta,\phi) + \frac{1}{\rho}\frac{\partial[\rho z_n(\rho)]}{\partial\rho}\underline{n}_{mn}(\theta,\phi)$$

The kind of the spherical Bessel function involved depends on the particular field we are considering. For plane waves, because its stationary behaviour, we would choose the spherical Bessel function of first kind, while for the scattered wave by a sphere, we would choose the spherical Hankel function of first type (second type if you use -*j*)

Scattering problem: buried sphere

- Incident plane wave
- Reflected plane wave
- Transmitted plane wave
- Scattered wave
- Internal wave
- Scattered-reflected wave
- Scattered-transmitted wave



Plane waves: spherical coordinates

$$\underline{E}(\underline{r}) = \underline{e}_{pol} e^{i\underline{k}\cdot\underline{r}} = \left(E_{\theta i}\underline{\theta}_{0i} + E_{\phi i}\underline{\phi}_{0i}\right) e^{i\underline{k}\cdot\underline{r}}$$

We get:

$$\underline{E}(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} a_{mn}\underline{M}_{mn}^{(1)}(\underline{r}) + b_{mn}\underline{N}_{mn}^{(1)}(\underline{r})$$

with:

$$a_{mn} = (-1)^m i^n \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \underline{e}_{pol} \cdot \underline{m}_{mn}^*(\theta_i, \phi_i)$$

$$b_{mn} = (-1)^m i^{n-1} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \underline{e}_{pol} \cdot \underline{n}_{mn}^*(\theta_i, \phi_i)$$

Incident and transmitted wave

$$\underline{E}_i(\underline{r}) = \underline{e}_{pol} e^{i\underline{k}\cdot\underline{r}} = \left(E_{\theta i}\underline{\theta}_{0i} + E_{\phi i}\underline{\phi}_{0i} \right) e^{i\underline{k}\cdot\underline{r}}$$

$$\underline{E}_t(\underline{r}) = e^{ik_2 h \cos\theta_t} \left(T_H^{12} E_{\theta i}\underline{\theta}_{0t} + T_E^{12} E_{\phi i}\underline{\phi}_{0t} \right) e^{i\underline{k}_t\cdot\underline{r}}$$

Hence:

$$\underline{E}_t(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} a_{mn}\underline{M}_{mn}^{(1)}(\underline{r}) + b_{mn}\underline{N}_{mn}^{(1)}(\underline{r})$$

Incident and transmitted wave

$$\underline{E}_i(\underline{r}) = \underline{e}_{pol} e^{i\underline{k}\cdot\underline{r}} = \left(E_{\theta i}\underline{\theta}_{0i} + E_{\phi i}\underline{\phi}_{0i} \right) e^{i\underline{k}\cdot\underline{r}}$$

$$\underline{E}_t(\underline{r}) = e^{ik_2 h\cos\theta_t} \left(T_H^{12} E_{\theta i}\underline{\theta}_{0t} + T_E^{12} E_{\phi i}\underline{\phi}_{0t} \right) e^{i\underline{k}_t\cdot\underline{r}}$$

Hence:

The coefficients must computed with the right polarization constant!

$$\underline{E}_t(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} a_{mn}\underline{M}_{mn}^{(1)}(\underline{r}) + b_{mn}\underline{N}_{mn}^{(1)}(\underline{r})$$

Scattered wave

$$\underline{E}_s(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} e_{mn} \underline{M}_{mn}^{(3)}(\underline{r}) + f_{mn} \underline{N}_{mn}^{(3)}(\underline{r})$$

As usual, the scattering problem reduces to found the expansions' coefficients

The scattered wave interacts with the planar interface

We must found the plane-wave spectrum of the vector functions

Back to the functions

Thanks to the beautiful properties of the vector tesseral functions, we have, for free, such plane-wave expansions:

$$\underline{M}_{mn}^{(3)} = \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} \underline{m}_{mn}(\alpha,\beta)e^{i\underline{k}\cdot\underline{r}}\sin\alpha d\alpha d\beta$$

$$\underline{N}_{mn}^{(3)} = \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} i\underline{n}_{mn}(\alpha,\beta)e^{i\underline{k}\cdot\underline{r}}\sin\alpha d\alpha d\beta$$

Scattered-reflected

Considering the reflection of each plane wave of the spectrum of the scattered wave and by an integral superposition, we get:

$$\underline{E}_{sr}(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \frac{i^{-n}}{2\pi} \left\{ e_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Re}^{mn} e^{i\underline{k}_r \cdot \underline{r}} d\alpha d\beta + f_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Rf}^{mn} e^{i\underline{k}_r \cdot \underline{r}} d\alpha d\beta \right\}$$

With:

$$\underline{E}_{Re}^{mn} = e^{im\beta} \sin\alpha \left[iR_H^{21} \pi_{mn}(\alpha) \underline{\alpha}_{0r} - R_E^{21} \tau_{mn}(\alpha) \underline{\beta}_{0r} \right] e^{2ik_2 h \cos\alpha}$$

$$\underline{E}_{Rf}^{mn} = e^{im\beta} \sin\alpha \left[R_H^{21} \tau_{mn}(\alpha) \underline{\alpha}_{0r} + iR_E^{21} \pi_{mn}(\alpha) \underline{\beta}_{0r} \right] e^{2ik_2 h \cos\alpha}$$

Scattered-reflected wave

Considering the reflection of each plane wave of the spectrum of the scattered wave and by an integral superposition, we get:

$$\underline{E}_{sr}(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \frac{i^{-n}}{2\pi} \left\{ e_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Re}^{mn} e^{i\underline{k}_r \cdot \underline{r}} d\alpha d\beta + f_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Rf}^{mn} e^{i\underline{k}_r \cdot \underline{r}} d\alpha d\beta \right\}$$

With:

$$\underline{E}_{Re}^{mn} = e^{im\beta} \sin\alpha \left[iR_H^{21} \pi_{mn}(\alpha) \underline{\alpha}_{0r} - R_E^{21} \tau_{mn}(\alpha) \underline{\beta}_{0r} \right] e^{2ik_2 h \cos\alpha}$$

$$\underline{E}_{Rf}^{mn} = e^{im\beta} \sin\alpha \left[R_H^{21} \tau_{mn}(\alpha) \underline{\alpha}_{0r} + iR_E^{21} \pi_{mn}(\alpha) \underline{\beta}_{0r} \right] e^{2ik_2 h \cos\alpha}$$

We lost the vector tesseral functions, because the two polarizations are differently reflected!

Scattered-transmitted wave

Similarly, for the scattered-transmitted wave:

$$\underline{E}_{st}(\underline{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \frac{i^{-n}}{2\pi} \left\{ e_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Te}^{mn} e^{i\underline{k}_t \cdot \underline{r}} d\alpha d\beta + f_{mn} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} \underline{E}_{Tf}^{mn} e^{i\underline{k}_t \cdot \underline{r}} d\alpha d\beta \right\}$$

With:

$$\underline{E}_{Te}^{mn} = e^{im\beta} \sin\alpha_t \left[iT_H^{21} \pi_{mn}(\alpha_t) \underline{\alpha}_{0t} - T_E^{21} \tau_{mn}(\alpha_t) \underline{\beta}_{0t} \right] e^{-ik_2 h \cos\alpha} e^{-ik_1 h \cos\alpha_t}$$

$$\underline{E}_{Tf}^{mn} = e^{im\beta} \sin\alpha_t \left[T_H^{21} \tau_{mn}(\alpha_t) \underline{\alpha}_{0t} + iT_E^{21} \pi_{mn}(\alpha_t) \underline{\beta}_{0t} \right] e^{-ik_2 h \cos\alpha} e^{-ik_1 h \cos\alpha_t}$$

Internal wave to the sphere

Inside the sphere we have another unknown field. It is a stationary wave, so we can obtain it as an expansion with the spherical Bessel functions of the first kind:

$$\underline{E}_{sp}\left(\underline{r}\right) = \sum_{n=1}^{+\infty} \sum_{m=-n}^{n} v_{mn} \underline{M}_{mn}^{(1)}\left(\underline{r}\right) + w_{mn} \underline{N}_{mn}^{(1)}\left(\underline{r}\right)$$

The coefficients of this expansion are other unknowns of our scattering problem.

Boundary conditions

If we consider the simple case of a PEC buried sphere:

$$\left(\underline{E}_t + \underline{E}_s + \underline{E}_{sr}\right) \times \underline{r}_0 \,\Big|_{r=a} = 0$$

Else, if the sphere is dielectric:

$$\left(\underline{E}_t + \underline{E}_s + \underline{E}_{sr} - \underline{E}_{sp}\right) \times \underline{r}_0 \,\Big|_{r=a} = 0$$

$$\left[\nabla \times \left(\underline{E}_t + \underline{E}_s + \underline{E}_{sr} - \underline{E}_{sp}\right)\right] \times \underline{r}_0 \,\Big|_{r=a} = 0$$

Next problem

The boundary conditions are on the surface of the sphere

However, the scattered-reflected field is expressed as

a plane-wave expansion!

We need to expand again the elementary plane waves within the integral

in a series of spherical harmonics:

$$\underline{E}_{sr}(\underline{r}) = \sum_{q=1}^{+\infty} \sum_{p=-q}^{q} \left[\underline{M}_{pq}^{(1)}(\underline{r}) \left(\sum_{n=1}^{+\infty} \sum_{m=-n}^{-n} e_{mn} C_{mn}^{pq} + f_{mn} D_{mn}^{pq} \right) + \underline{N}_{pq}^{(1)}(\underline{r}) \left(\sum_{n=1}^{+\infty} \sum_{m=-n}^{-n} e_{mn} G_{mn}^{pq} + f_{mn} H_{mn}^{pq} \right) \right]$$

Next problem (2)

$$\underline{E}_{sr}(\underline{r}) = \sum_{q=1}^{+\infty} \sum_{p=-q}^{q} \left[\underline{M}_{pq}^{(1)}(\underline{r}) \left(\sum_{n=1}^{+\infty} \sum_{m=-n}^{-n} e_{mn} C_{mn}^{pq} + f_{mn} D_{mn}^{pq} \right) + \underline{N}_{pq}^{(1)}(\underline{r}) \left(\sum_{n=1}^{+\infty} \sum_{m=-n}^{-n} e_{mn} G_{mn}^{pq} + f_{mn} H_{mn}^{pq} \right) \right]$$

The capital C, D, G, and H contain the integrals and they are known quantities

$$C_{pq}^{mn} = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} c_{pq}^{mn}(\alpha,\beta) d\alpha d\beta$$

$$D_{pq}^{mn} = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} d_{pq}^{mn}(\alpha,\beta) d\alpha d\beta$$

$$G_{pq}^{mn} = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} g_{pq}^{mn}(\alpha,\beta) d\alpha d\beta$$

$$H_{pq}^{mn} = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}-i\infty} h_{pq}^{mn}(\alpha,\beta) d\alpha d\beta$$

The constants c and d, and, g and h, are the expansion coefficients relevant to the elementary plane waves in the integrals multiplied by coefficients e_{mn} and f_{mn}, respectively

Solution

Imposition of the boundary conditions drive us to the following matrix form:

$$\begin{pmatrix} e_{mn} \\ f_{mn} \\ v_{mn} \\ w_{mn} \end{pmatrix} = \begin{pmatrix} \delta_{mp}\delta_{nq}\dfrac{h_n^{(1)}(k_2 a)}{j_n(k_2 a)} + C_{mn}^{pq} & G_{mn}^{pq} & \delta_{mp}\delta_{nq}\dfrac{j_n(k_3 a)}{j_n(k_2 a)} & 0 \\ \delta_{mp}\delta_{nq}\dfrac{\dot{h}_n^{(1)}(k_2 a)}{\dot{j}_n(k_2 a)} + C_{mn}^{pq} & G_{mn}^{pq} & \delta_{mp}\delta_{nq}\dfrac{\dot{j}_n(k_3 a)}{\dot{j}_n(k_2 a)} & 0 \\ D_{mn}^{pq} & \delta_{mp}\delta_{nq}\dfrac{\dot{j}_n(k_2 a)}{\dot{h}_n^{(1)}(k_2 a)} + H_{mn}^{pq} & 0 & \delta_{mp}\delta_{nq}\dfrac{\dot{j}_n(k_3 a)}{\dot{j}_n(k_2 a)} \\ D_{mn}^{pq} & \delta_{mp}\delta_{nq}\dfrac{j_n(k_2 a)}{h_n^{(1)}(k_2 a)} + H_{mn}^{pq} & 0 & \delta_{mp}\delta_{nq}\dfrac{j_n(k_3 a)}{j_n(k_2 a)} \end{pmatrix}^{-1} \begin{pmatrix} -a_{pq} \\ -a_{pq} \\ -b_{pq} \\ -b_{pq} \end{pmatrix}$$

The solution of the system has been obtain thanks to

a LU decomposition of the matrix

Solution (2)

For what concerns the integrals, they present an integration path on the complex plane, being angular spectra:

$$C_{mn}^{pq} = i^{q-n}(-1)^{-q} \frac{2q+1}{q(q+1)} \frac{(q-p)!}{(q+p)!} \delta_{pm} \int_{0}^{\pi/2-i\infty} \left[R_H^{21} \pi_n^m(\cos\alpha_r) \pi_p^q(\cos\alpha_r) - R_E^{21} \tau_n^m(\cos\alpha_r) \tau_p^q(\cos\alpha_r) \right] \sin\alpha_r e^{2ihk_2\cos\alpha_r} d\alpha_r d\beta_r$$

We written the integral as a sum of two integrals with reals integration paths

$$I = \int_0^{\frac{\pi}{2}-i\infty} f(\cos\alpha) e^{i2q\cos\alpha} \sin\alpha \, d\alpha$$

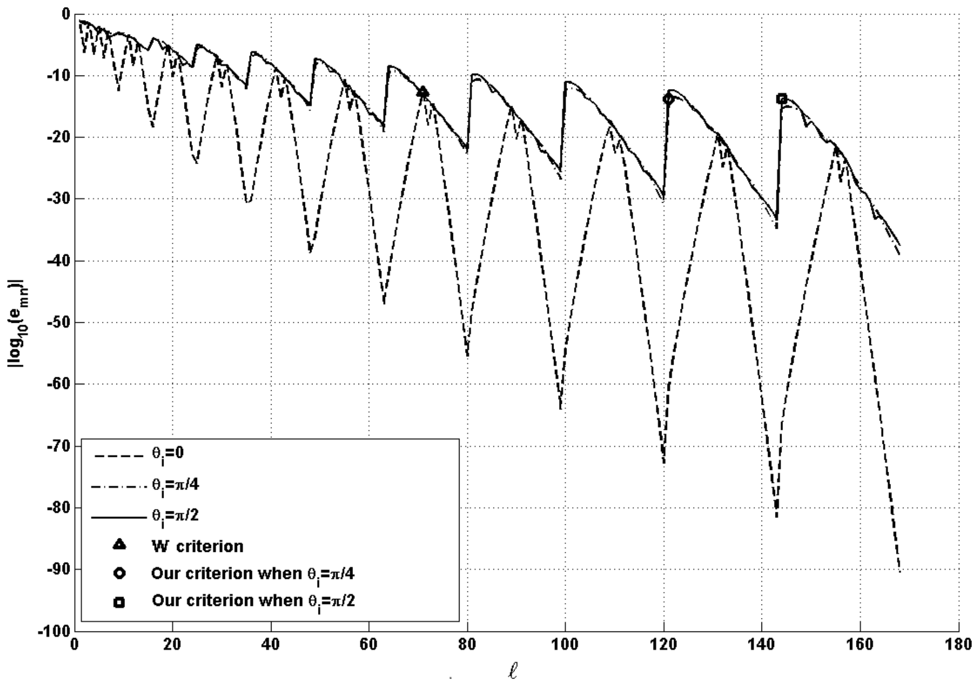
$$I_R = \int_0^{\frac{\pi}{2}} f(\cos\alpha) e^{i2q\cos\alpha} \sin\alpha \, d\alpha$$

$$I_I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}-i\infty} f(\cos\alpha) e^{i2q\cos\alpha} \sin\alpha \, d\alpha \longrightarrow I_I = \int_0^{+\infty} f(iu) e^{-2qu} \, du$$

In this way, both the integrals are computable with a Gauss-Legendre quadrature algorithm

Truncation criteria

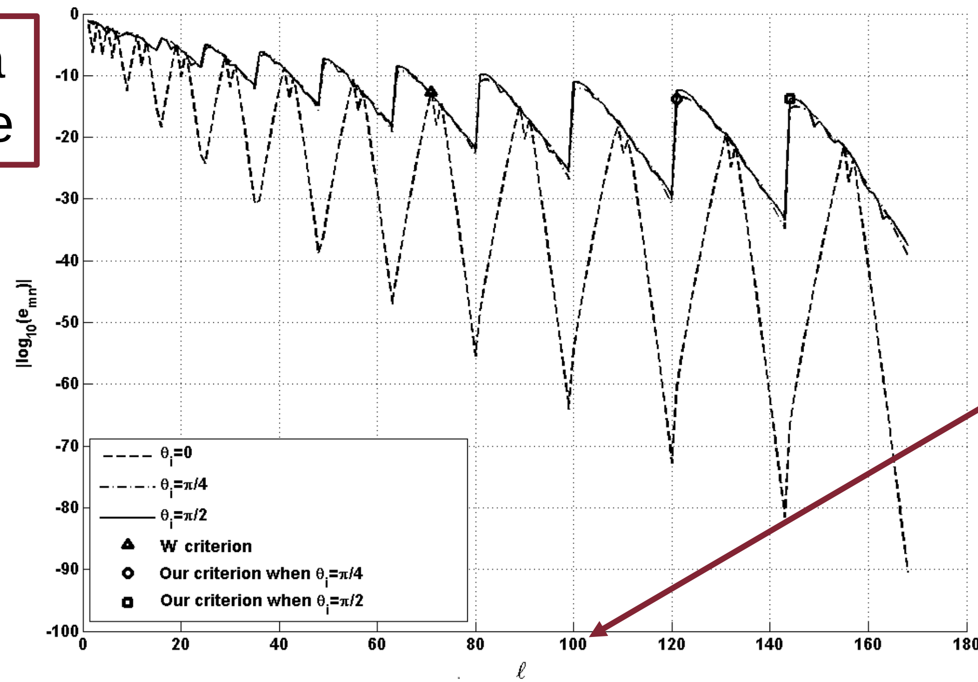
By a careful analysis of the convergence behaviour of the coefficients, we noted that the convergence does not depend only on the sphere's radius, but it depends also on the incident angle.



Truncation criteria (2)

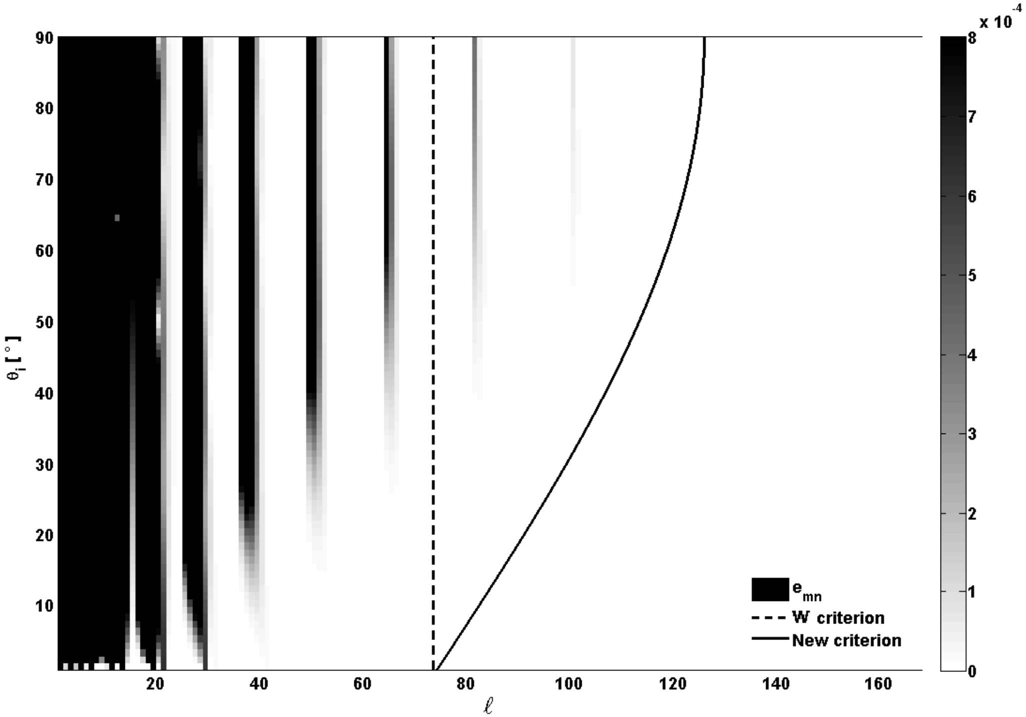
By a careful analysis of the convergence behaviour of the coefficients, we noted that the convergence does not depend only on the sphere's radius, but it depends also on the incident angle.

Coefficients in a logarithmic scale



$l = n(n+1) + m$

Truncation criteria (3)



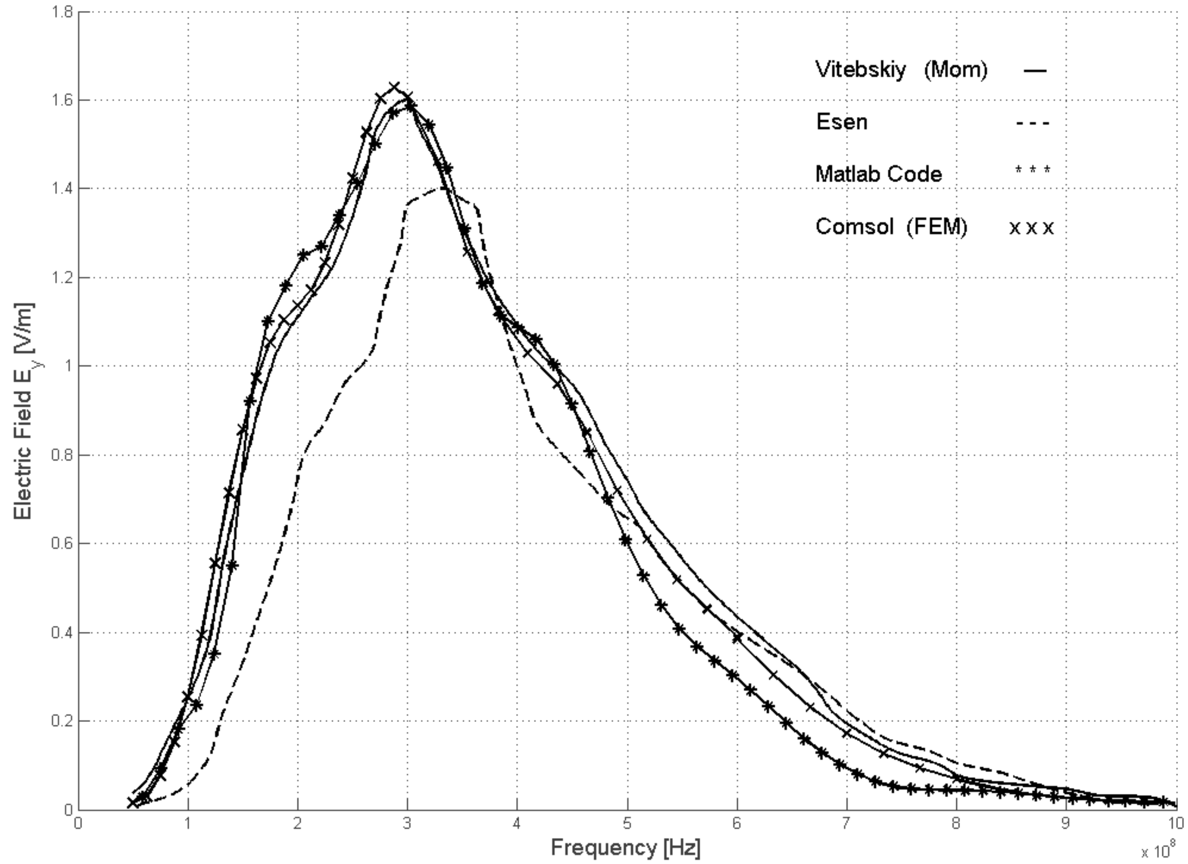
$$\boxed{\mathrm{N} = x + 4x^{1/3} + 2}$$

$$x = k_2 a$$ Wiscombe Criterion

$$x = k_2 a\left(\sin \vartheta_i + 1\right)$$ New Criterion

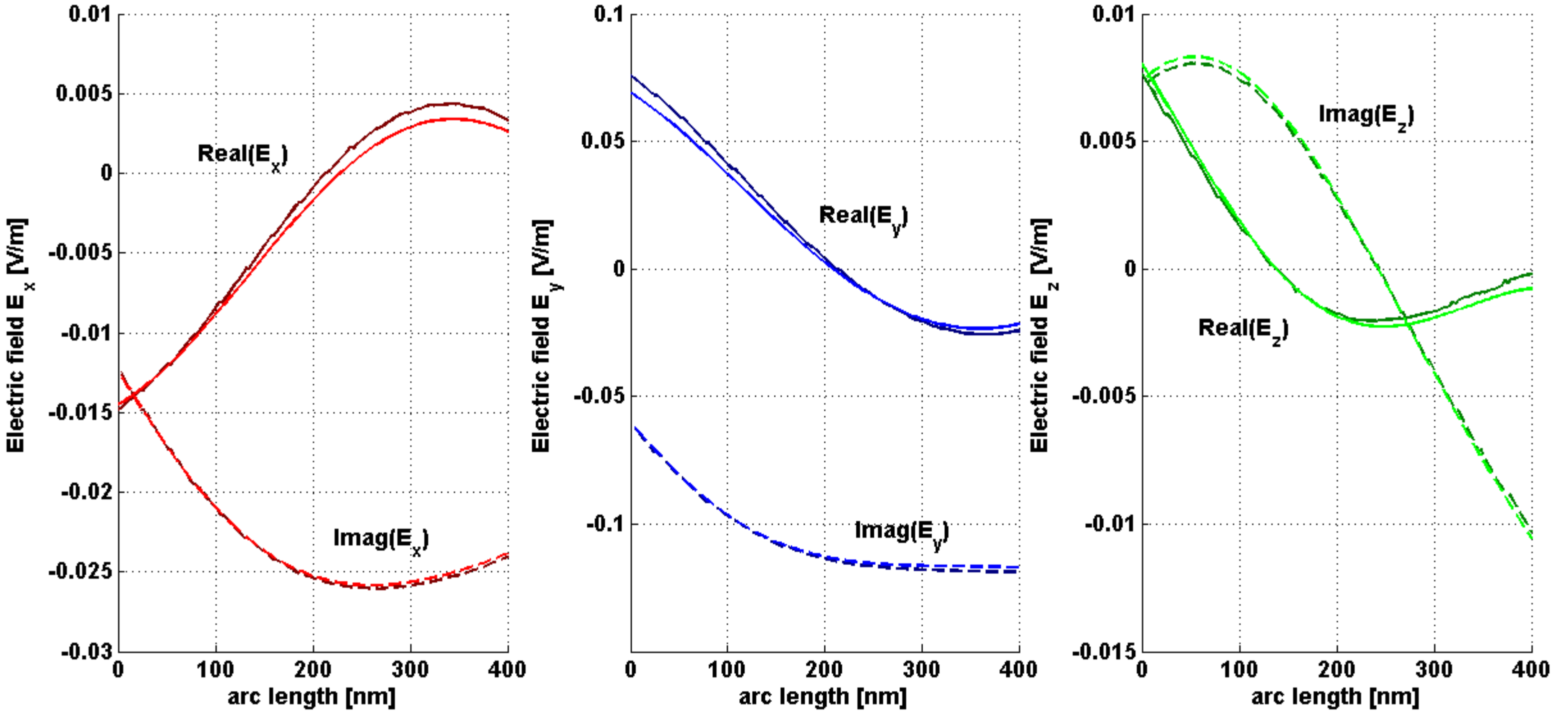
Results

Validation by comparisons with the literature and with a FEM method (normal incidence)



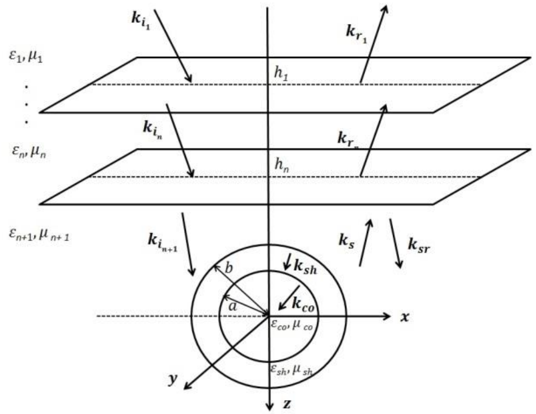
Results (2)

Validation by comparisons with a FEM method
(oblique incidence)

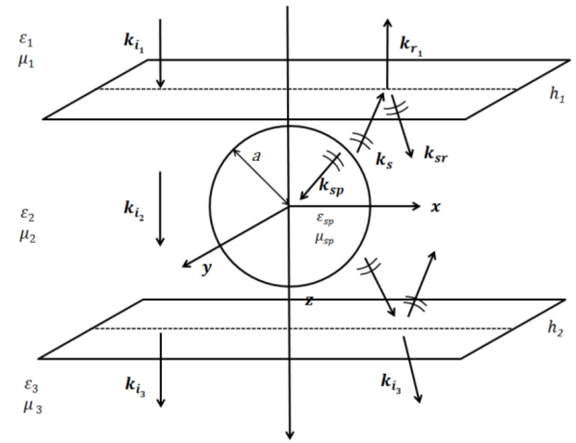


Extensions

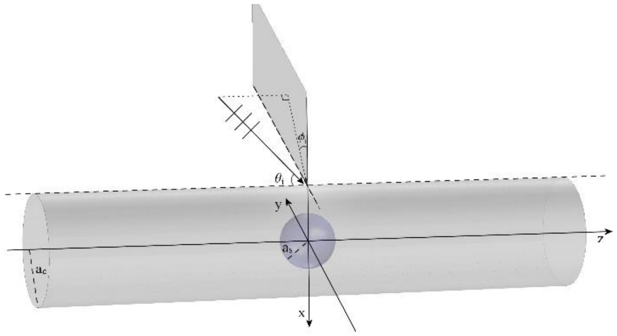
Sphere below a stratified media



Sphere inside a stratified media



Sphere inside a cylinder!!! (Coming soon)



Conclusions

- We saw how the vector wave equation can be solved with the solution of the scalar wave equation
- The spherical harmonics have been obtained and written in a simple form
- The problem of the scattering by a dielectric sphere buried in a ground has been faced
- The numerical aspects of the problem have been considered
- Validations through the literature and other numerical methods have been presented