Time domain analysis (of the GPR transmitted field)

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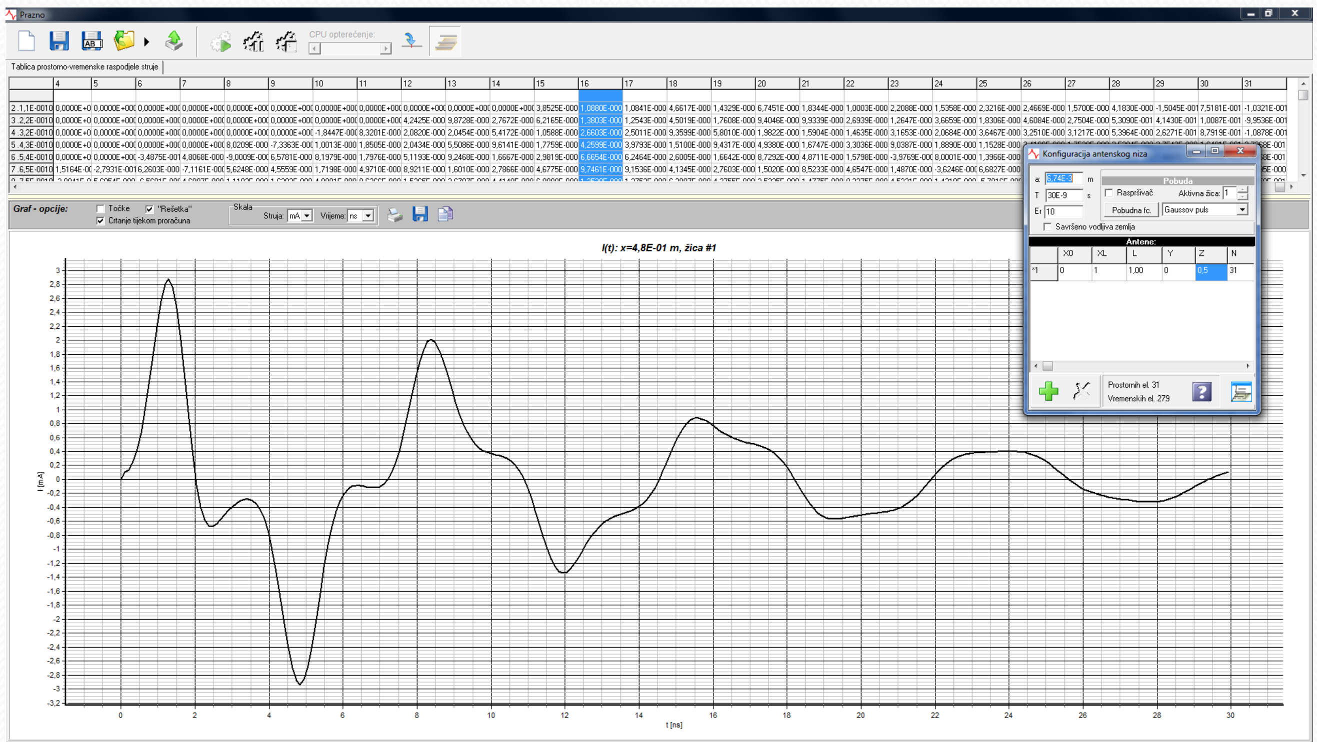
SuzanaTD

Introduction

SuzanaTD

- Introduction to software SuzanaTD



Integral equations

…and methods for their solution

Pocklington equation (1897.)

Frequency domain

$$E_x^{exc}(\omega) = -\frac{1}{j4\pi\omega\varepsilon_{eff}}\int_0^L \left(\frac{\partial^2}{\partial x^2}-\gamma^2\right) I(x',\omega)\, g(x,x')\, dx'$$

Time domain

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma\right)E_x^{exc}(t) = -\left(\frac{\partial^2}{\partial x^2}-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\int_0^L \frac{I\left(x',t-\frac{R}{v}\right)}{4\pi R}\, dx'$$

Hallen equation (1938.)

Frequency domain

$$\int_0^L I(x',\omega)\,g(x,x')\,dx' = C\cos(kx) + B\sin(kx) - \frac{j4\pi}{Z_0}\int_0^L E_x^{exc}(x',\omega)\sin\left[k(x-x')\right]dx'$$

Time domain

$$\int_0^L \frac{I\left(x',t-\frac{R}{c}\right)}{4\pi R}dx' = F_0\left(t-\frac{x}{c}\right) + F_L\left(t-\frac{L-x}{c}\right) + \frac{1}{2Z_0}\int_0^L E_x^{exc}\left(x',t-\frac{|x-x'|}{c}\right)dx'$$

Methods of solving integral equations

- Numerical solutions
  - Complex geometries
  - Wide range of parameters
- Analytical solutions
  - Canonical geometries
  - Limited parameters
  - Use of approximations
  - Procedure control
  - Simple solutions

Analytical solution

Time domain approximations and procedure

Time domain approximations – thin-wire approximation

$$\vec{A}(\vec{r},t)=\mu\int_{V'}\vec{J}\left(\vec{r}\,',t-\frac{R}{v}\right)\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{4\pi R}dV'$$

$$A_x(x,t)=\frac{\mu}{4\pi}\int_0^L I\left(x',t-\frac{R}{v}\right)\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx'$$

$$\left(\mu\varepsilon\frac{\partial}{\partial t}+\mu\sigma\right)\vec{E}=\nabla^2\vec{A}-\mu\sigma\frac{\partial\vec{A}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\left(\mu\varepsilon\frac{\partial}{\partial t}+\mu\sigma\right)E_x^{tr}=-\left(\frac{\partial^2 A_x}{\partial x^2}-\mu\sigma\frac{\partial A_x}{\partial t}-\mu\varepsilon\frac{\partial^2 A_x}{\partial t^2}\right)$$

Time domain approximations – reflection coefficient

$$\Gamma_{ref} = \frac{\dfrac{1}{\varepsilon_r + \dfrac{\sigma}{s\varepsilon_0}}\cos\theta - \sqrt{\dfrac{1}{\varepsilon_r + \dfrac{\sigma}{s\varepsilon_0}} - \sin^2\theta}}{\dfrac{1}{\varepsilon_r + \dfrac{\sigma}{s\varepsilon_0}}\cos\theta + \sqrt{\dfrac{1}{\varepsilon_r + \dfrac{\sigma}{s\varepsilon_0}} - \sin^2\theta}}$$

$$\Gamma_{ref}^{MIT}(s) = -\frac{s\tau_1 + 1}{s\tau_2 + 1}$$

$$\Gamma_{ref}^{MIT}(t) = -\left[\frac{\tau_1}{\tau_2}\delta(t) + \frac{1}{\tau_2}\left(1 - \frac{\tau_1}{\tau_2}\right)e^{-\frac{t}{\tau_2}}\right]$$

Time domain approximations – integral function

$$\int_0^L I\left(x',t-\frac{R}{v}\right)\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx' = \int_0^L \left[I\left(x,t-\frac{a}{v}\right) + I\left(x',t-\frac{R}{v}\right) - I\left(x,t-\frac{a}{v}\right)\right]\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx'$$

$$I\left(x',t-\frac{R}{v}\right) - I\left(x,t-\frac{a}{v}\right) \equiv 0$$

$$\int_0^L I\left(x',t-\frac{R}{v}\right)\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx' = I\left(x,t-\frac{a}{v}\right)\int_0^L \frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx'$$

Laplace transform

$$\left(\mu\varepsilon\frac{\partial}{\partial t}+\mu\sigma\right)E_x^{tr}\left(t\right)=-\left(\frac{\partial^2}{\partial x^2}-\mu\sigma\frac{\partial}{\partial t}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\right)$$

$$\bullet\left[\frac{\mu}{4\pi}I\left(x,t-\frac{a}{v}\right)\int_0^L\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx'-\frac{\mu}{4\pi}\int_0^t\Gamma_{ref}^{MIT}\left(\tau\right)I\left(x,t-\frac{a}{v}-\tau\right)\int_0^L\frac{e^{-\frac{1}{\tau_g}\frac{R^*}{v}}}{R^*}dx'd\tau\right]$$

$$\left(\mu\varepsilon s+\mu\sigma\right)E_x^{tr}\left(s\right)=-\frac{\mu}{4\pi}\left(\frac{\partial^2}{\partial x^2}-\mu\sigma s-\mu\varepsilon s^2\right)$$

$$\bullet I\left(x,s\right)e^{-\frac{a}{v}s}\left[\int_0^L\frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx'-\Gamma_{ref}^{MIT}\left(s\right)\int_0^L\frac{e^{-\frac{1}{\tau_g}\frac{R^*}{v}}}{R^*}dx'\right]$$

Second order differential equation

$$\int_0^L \frac{e^{-\frac{1}{\tau_g}\frac{R}{v}}}{R}dx' - \Gamma_{ref}^{MIT}(s)\int_0^L \frac{e^{-\frac{1}{\tau_g}\frac{R^*}{v}}}{R^*}dx' = 2\left(\ln\frac{L}{a} + \frac{s\tau_1+1}{s\tau_2+1}\ln\frac{L}{2d}\right)$$

$$\frac{\partial^2 I(x,s)}{\partial x^2} - \gamma^2 I(x,s) = -\frac{4\pi}{\mu s \Psi(s)} e^{\frac{a}{v}s}\gamma^2 E_x^{tr}(s)$$

$$I(x,s) = \frac{4\pi e^{\frac{a}{v}s}}{\mu s \Psi(s)} E_x^{tr}(s)\left[1 - \frac{\cosh\left(\gamma\left(\frac{L}{2}-x\right)\right)}{\cosh\left(\gamma\frac{L}{2}\right)}\right]$$

Cauchy residue theorem

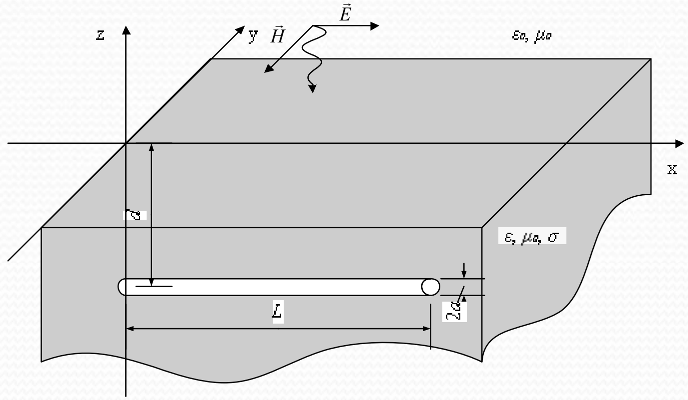
$$f(t) = \lim_{y\to\infty} \frac{1}{j2\pi} \int_{x-jy}^{x+jy} e^{ts} F(s)\, ds = \sum_{k=1}^{n} \mathrm{Res}(s_k)$$

$$I(x,t) = \frac{4\pi}{\mu} \left\{ \begin{array}{l} R(s_\Psi)\left[1 - \dfrac{\cosh\left(\gamma_\Psi\left(\dfrac{L}{2} - x\right)\right)}{\cosh\left(\gamma_\Psi \dfrac{L}{2}\right)}\right] e^{\left(t+\frac{a}{v}\right)s_\Psi} \\ -\dfrac{\pi}{\mu\varepsilon L^2} \displaystyle\sum_{n=1}^{\infty} \dfrac{2n-1}{\pm\sqrt{b^2 - 4c_n}\, s_{1,2n} \Psi(s_{1,2n})} \sin\dfrac{(2n-1)\pi x}{L} e^{\left(t+\frac{a}{v}\right)s_{1,2n}} \end{array} \right\}$$

Analytical solutions

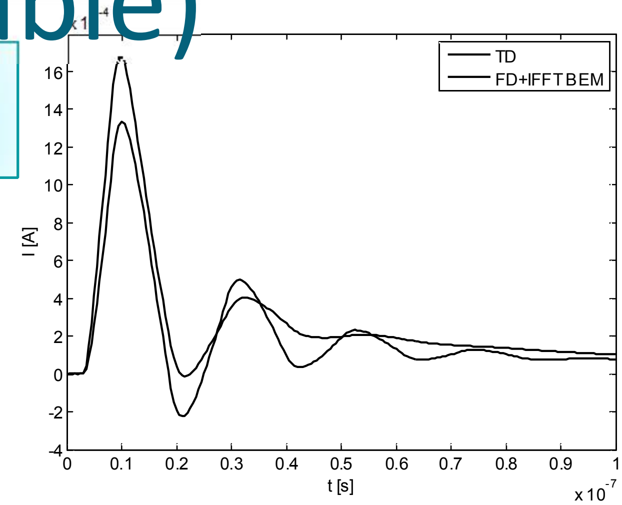
Examples

Wire below ground (telecommunication cable)

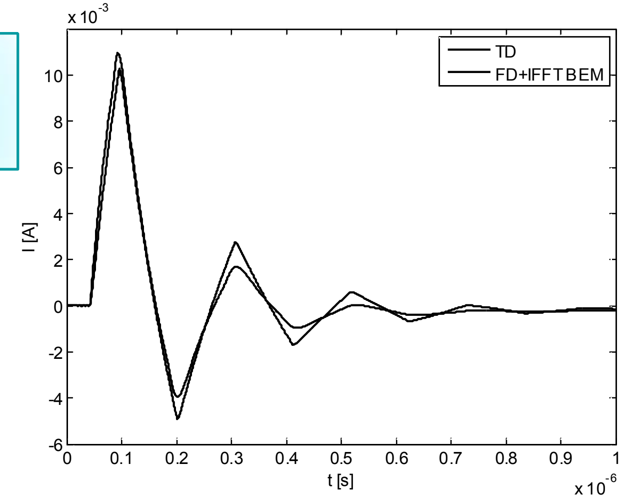


$$I(x,t)=\frac{4\pi}{\mu}\left\{ R(s_\Psi)\left[1-\frac{\cosh\left(\gamma_\Psi\left(\frac{L}{2}-x\right)\right)}{\cosh\left(\gamma_\Psi\frac{L}{2}\right)}\right]e^{\left(t+\frac{a}{v}\right)s_\Psi} - \frac{\pi}{\mu\varepsilon L^2}\sum_{n=1}^{\infty}\frac{2n-1}{\pm\sqrt{b^2-4c_n s_{1,2n}\Psi(s_{1,2n})}}\sin\frac{(2n-1)\pi x}{L}e^{\left(t+\frac{a}{v}\right)s_{1,2n}}\right\}$$

L=1 m
σ=10 mS/m



L=10 m
σ=1 mS/m



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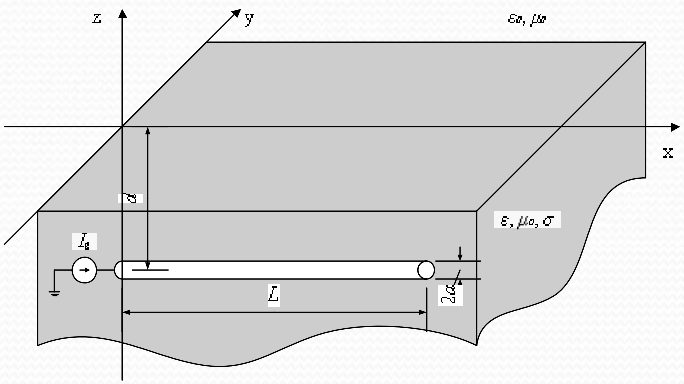
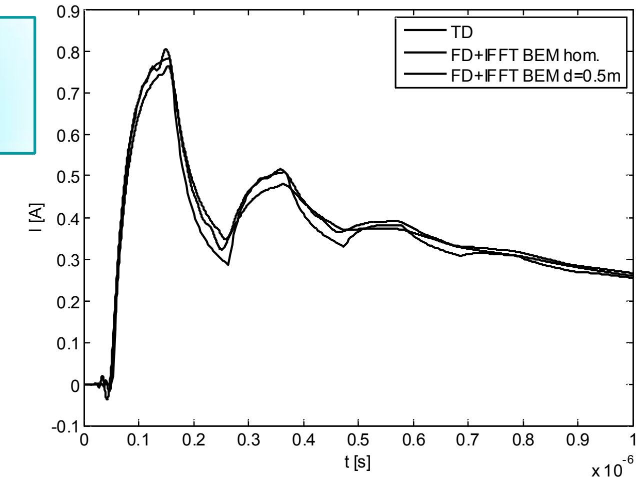
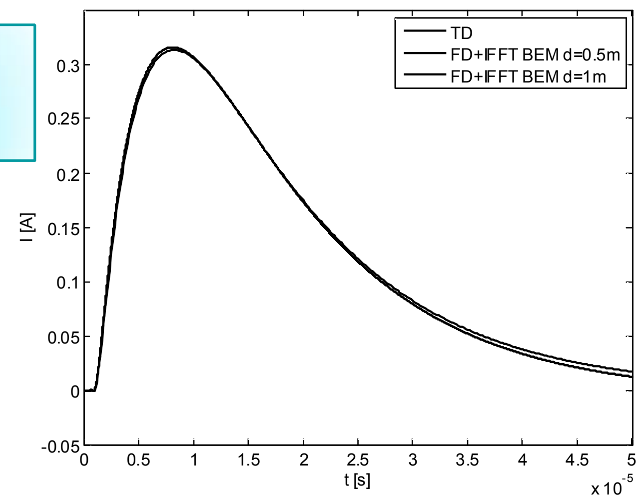
Wire below ground (grounding electrode)



L=10 m
σ=1 mS/m

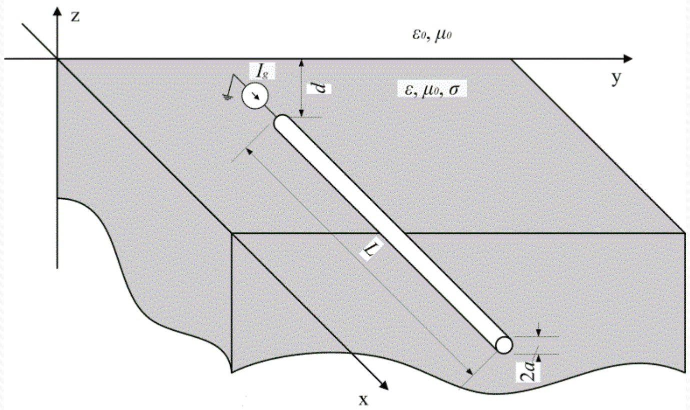


L=200 m
σ=1 mS/m



$$I(x,t)=\frac{2\pi}{\mu\varepsilon L^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{\pm\sqrt{b^2-4c_n}}\sin\frac{n\pi(L-x)}{L}e^{ts_{1,2n}}$$

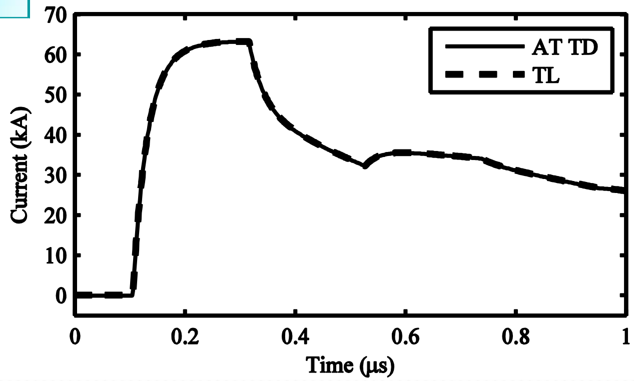
Wire below ground (grounding electrode)



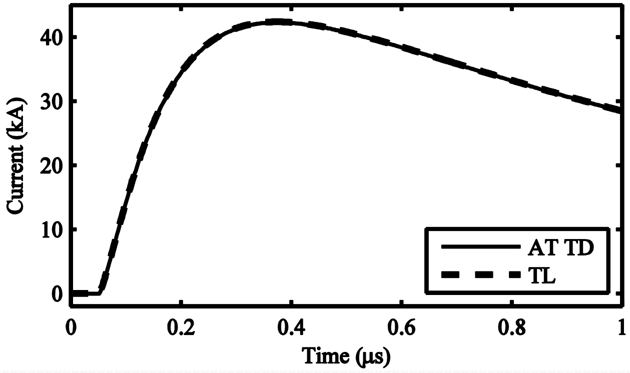
$$I(x,t) = \frac{2\pi I_0}{\mu\varepsilon L^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\pm\sqrt{b^2 - 4c_n}} \sin\frac{n\pi(L-x)}{L}$$

$$\cdot\left(\frac{e^{s_{1,2n}t} - e^{-\alpha t}}{s_{1,2n} + \alpha} - \frac{e^{s_{1,2n}t} - e^{-\beta t}}{s_{1,2n} + \beta} \right)$$

L=20 m
σ=1 mS/m

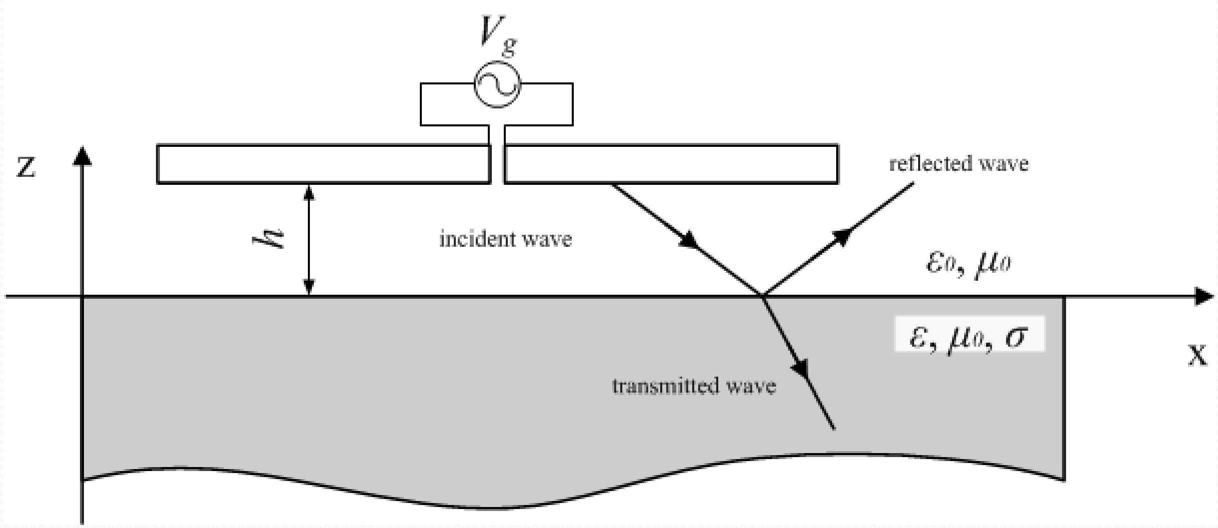


L=10 m
σ=10 mS/m



Transmitted field - formulation

Transmitting GPR dipole antenna



Hallen integral equation

$$\int_0^L \frac{I\left(x',t-R_a/c\right)}{4\pi R_a}dx' - \int_{-\infty}^{t}\int_0^L r(\theta,\tau)\frac{I\left(x',t-R_a^*/c-\tau\right)}{4\pi R_a^*}dx'd\tau$$

$$= \frac{1}{2Z_0}\int_0^L E_x^{inc}\left(x',t-\frac{|x-x'|}{c}\right)dx' + F_0\left(t-\frac{x}{c}\right) + F_L\left(t-\frac{L-x}{c}\right)$$

Hallen integral equation

$$r(\theta,t) = K\delta(t) + \frac{4\beta}{1-\beta^2}\frac{e^{-\alpha t}}{t}\sum_{n=1}^{\infty}(-1)^{n+1} nK^n I_n(\alpha t)$$

Reflection coefficient

$$K = \frac{1-\beta}{1+\beta};\ \beta = \frac{\sqrt{\varepsilon_r - \sin^2\theta}}{\varepsilon_r \cos\theta};\ \alpha = \frac{\sigma}{2\varepsilon};\ \theta = arctg\frac{|x-x'|}{2h}$$

Total reflected field

Total field in the upper medium

$$E_x^{tot}(r,t) = -\frac{\mu_0}{4\pi}\left[\int_0^L \frac{\partial I\left(x',t-R_d/c\right)}{\partial t}\frac{1}{R_d}dx' - \int_{-\infty}^{t}\int_0^L \Gamma_{ref}^{MIT}(\tau)\frac{\partial I\left(x',t-R_d^*/c-\tau\right)}{\partial t}\frac{1}{R_d^*}dx'd\tau\right]$$

$$\Gamma_{ref}^{MIT}(t) = \frac{\tau_1}{\tau_2}\delta(t) + \frac{1}{\tau_2}\left(1-\frac{\tau_1}{\tau_2}\right)e^{-\frac{t}{\tau_2}}$$

Reflection coefficient

$$\tau_1 = \frac{\varepsilon_0\left(\varepsilon_r - 1\right)}{\sigma} \qquad \tau_2 = \frac{\varepsilon_0\left(\varepsilon_r + 1\right)}{\sigma}$$

Boundary element formalism

$$E_{x,i}^{tot}(r,t) = -\frac{\mu_0}{4\pi}\left[\int_{\Delta l_i} \frac{I_{i1}^{m+1} - I_{i1}^{m}}{\Delta t} f_{i1}(x') \frac{1}{R_d} dx' + \int_{\Delta l_i} \frac{I_{i2}^{m+1} - I_{i2}^{m}}{\Delta t} f_{i2}(x') \frac{1}{R_d} dx'\right]$$

$$-\frac{\mu_0}{4\pi}\Gamma_{ref}^{MIT}\left[\int_{\Delta l_i} \frac{I_{i1}^{m+1} - I_{i1}^{m}}{\Delta t} f_{i1}(x') \frac{1}{R_d^*} dx' + \int_{\Delta l_i} \frac{I_{i2}^{m+1} - I_{i2}^{m}}{\Delta t} f_{i2}(x') \frac{1}{R_d^*} dx'\right]$$

$$I(x',t') = \{f\}^T \{I(t')\}$$

$$f_r(x') = \frac{x_{r+1} - x'}{x_{r+1} - x_r}; \; f_{r+1}(x') = \frac{x' - x_r}{x_{r+1} - x_r}$$

Spatial and temporal discretization

Transmitted field

Transmitted field in the lower medium

$$E_x^{tr}(r,t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{t}\int_{0}^{L}\Gamma_{tr}^{MIT}(\tau)\frac{\partial I\left(x',t-R''/v-\tau\right)}{\partial t}\frac{e^{-\frac{1}{\tau_g}\frac{R''}{v}}}{R''}dx'd\tau$$

$$\Gamma_{tr}^{MIT}(t) = \frac{\tau_3}{\tau_2}\delta(t) + \frac{1}{\tau_2}\left(1-\frac{\tau_3}{\tau_2}\right)e^{-\frac{t}{\tau_2}}$$

Transmission coefficient

$$\tau_2 = \frac{\varepsilon_0\left(\varepsilon_r+1\right)}{\sigma} \qquad \tau_3 = \frac{2\varepsilon_0\varepsilon_r}{\sigma}$$

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Transmitted field in the lower medium – dielectric half-space

$$E_x^{tr}(r,t) = \frac{\mu_0}{4\pi} \int_0^L \Gamma_{tr}^{MIT} \frac{\partial I\left(x', t - R''/v\right)}{\partial t} \frac{1}{R''} dx'$$

$$\Gamma_{tr}^{MIT} = \frac{2\varepsilon_r}{\varepsilon_r + 1}$$

Transmission coefficient

Transmitted field in the lower medium – lossy half-space

$$E_x^{tr}\left(r,t\right)=-\frac{\mu}{4\pi}\frac{\partial}{\partial t}\int_{-\infty}^{t}\int_{0}^{L}\Gamma_{tr}\left(t-\tau\right)I\left(x',\tau-\frac{R''}{v}\right)\frac{e^{-\frac{1}{\tau_3}\frac{R''}{v}}}{R''}dx'd\tau$$

$$E_{xd}^{tr}\left(r,t\right)=-\frac{\mu_0}{4\pi}\int_{0}^{L}\frac{\tau_3}{\tau_2}\frac{\partial}{\partial t}I\left(x',t-\frac{R''}{v}\right)\frac{1}{R''}dx'$$

$$E_{xg}^{tr}\left(r,t\right)=-\frac{\mu}{4\pi}\int_{-\infty}^{t}\int_{0}^{L}\Gamma_{tr}^{d}\left(t-\tau\right)I\left(x',\tau-\frac{R''}{v}\right)\frac{e^{-\frac{1}{\tau_3}\frac{R''}{v}}}{R''}dx'd\tau$$

$$\Gamma_{tr}^{d}\left(t-\tau\right)=-\frac{1}{\tau_2^2}\left(2-\frac{\tau_3}{\tau_2}\right)e^{-\frac{t-\tau}{\tau_2}}$$

Boundary element formalism

$$I\left(x',t-\frac{R''}{v}\right)=\sum_{i=1}^{N_g} I_i\left(t-\frac{R''}{v}\right)f_i\left(x'\right)$$

Local expansion of the current

$$I\left(x',t'\right)=\left\{f\right\}^T\left\{I\left(t'\right)\right\}$$

$$f_r\left(x'\right)=\frac{x_{r+1}-x'}{x_{r+1}-x_r};\ f_{r+1}\left(x'\right)=\frac{x'-x_r}{x_{r+1}-x_r}$$

Linear approximation

Boundary element formalism

$$E_x^{tr} = \sum_{i=1}^{M} \left(E_{xd,i}^{tr} + E_{xg,i}^{tr} \right)$$

Spatial discretization

$$E_{xd,i}^{tr}\left(r,t\right) = \frac{\mu_0}{4\pi} \frac{\tau_3}{\tau_2} \left[\int_{\Delta l_i} \frac{\partial}{\partial t} \left\{ I \right\} \Big|_{t''=t-\frac{R''}{v}} \left\{ f' \right\}^T \frac{1}{R''} dx' \right]$$

$$E_{xg,i}^{tr} = -\frac{\mu}{4\pi} \int_{-\infty}^{t} \Gamma_{tr}^{d} \left(t-\tau \right) \sum_{i=1}^{N_g} I_i \left(\tau \right) \int_{\Delta x_i} \left\{ f' \right\}_i^T \frac{e^{-\frac{1}{\tau_3}\frac{R''}{v}}}{R''} dx' d\tau$$

Boundary element formalism

$$I_i\left(t-\frac{R''}{v}\right)=\sum_{k=1}^{N_t} I_i^k(t'')T^k(t''),\ t''=t-\frac{R''}{v}$$

Temporal discretization

$$T^k(t'')=\frac{t_{k+1}-t''}{\Delta t''},T^{k+1}(t'')=\frac{t''-t_k}{\Delta t''},\Delta t''=t_{k+1}-t_k$$

Boundary element formalism

$$E_{xd,i}^{tr}\left(r,t\right)=\frac{\mu_0}{4\pi}\frac{\tau_3}{\tau_2}\left[\begin{array}{l}\int\limits_{\Delta l_i}\frac{I_{i1}^{m+1}-I_{i1}^{m}}{\Delta t}f_{i1}\left(x'\right)\frac{1}{R''}dx'+\\ \int\limits_{\Delta l_i}\frac{I_{i2}^{m+1}-I_{i2}^{m}}{\Delta t}f_{i2}\left(x'\right)\frac{1}{R''}dx'\end{array}\right]$$

$$E_{xg,i}^{tr}=-\frac{\mu}{4\pi}\int\limits_{t_k}^{t_{k+1}}\Gamma_{tr}^{d}\left(t_k-\tau\right)\left[\left(I_{i1}^{m}\frac{\tau^{m+1}-\tau}{\Delta\tau}+I_{i1}^{m+1}\frac{\tau-\tau^{m}}{\Delta\tau}\right)\int\limits_{x_{i1}}^{x_{i2}}\frac{x_{i2}-x'}{\Delta x}\frac{e^{-\frac{1}{\tau_3}\frac{R''}{v}}}{R''}dx'\right.$$

$$\left.+\left(I_{i2}^{m}\frac{\tau^{m+1}-\tau}{\Delta\tau}+I_{i2}^{m+1}\frac{\tau-\tau^{m}}{\Delta\tau}\right)\int\limits_{x_{i1}}^{x_{i2}}\frac{x'-x_{i1}}{\Delta x}\frac{e^{-\frac{1}{\tau_3}\frac{R''}{v}}}{R''}dx'\right]d\tau$$

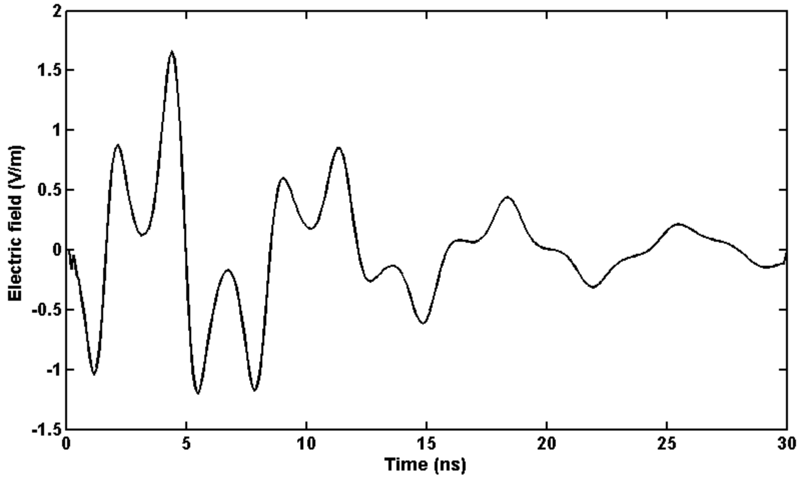
Numerical results

Parameters of the calculation

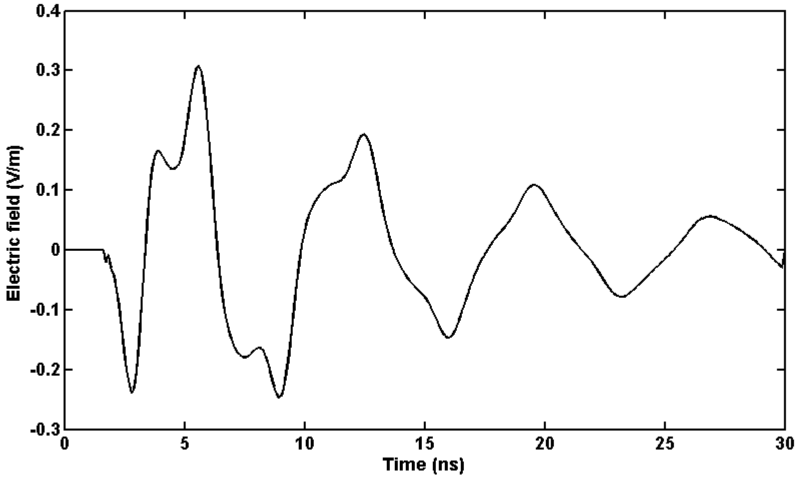
- Properties of the antenna
  - Length L=1 m
  - Radius a=6.74 mm
  - Height h=0.1 m
- Properties of the lower medium
  - Relative permittivity ε_r=10
  - Conductivity σ=1, 10 mS/m
- Properties of the excitation
  - Gaussian pulse V_0=1 V, g=1.5*10^9 s^{-1}, t_0=1.43 ns

Numerical results – upper medium

***h*=0.55 m**



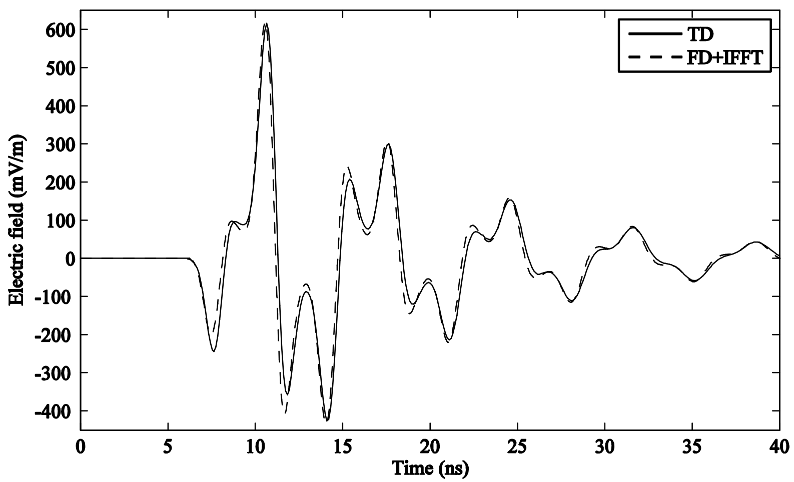
***h*=1 m**

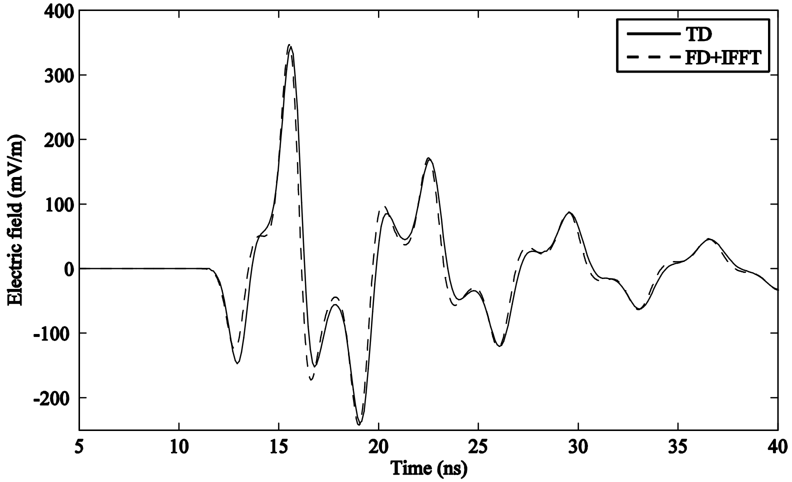


Numerical results – lower medium, σ=1 mS/m

***d*=0.5 m**

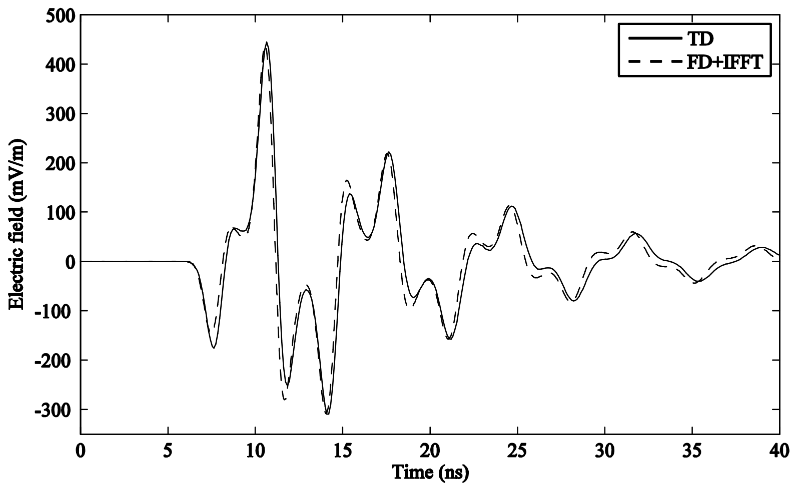
***d*=1 m**



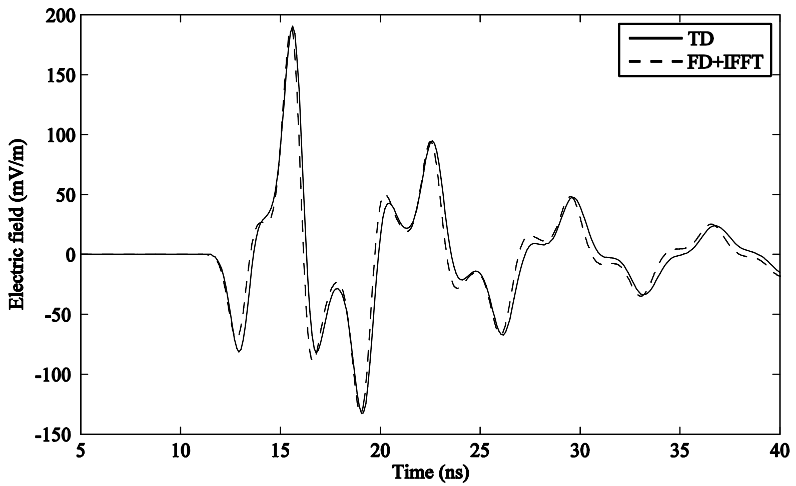


Numerical results – lower medium, σ=10 mS/m

***d*=0.5 m**



***d*=1 m**



Stochastic post-processing

Stochastic collocation technique

Polynomial approximation of the output E for N random parameters

$$Z = Z^0 + \hat{u}$$

Random parameter

$$E\left(Z^0, t\right) = \sum_{i=1}^{n} E_i\left(Z^0\right) L_i\left(t\right)$$

Lagrangian basis function set

$$L_i\left(t\right) = \prod_{j=1, j\neq i}^{n} \frac{\left(t - t_j\right)}{\left(t_j - t_i\right)}$$

Lagrange polynomial

Stochastic collocation technique

$$\mu = \left\langle E\left(Z^0, t\right)\right\rangle = \int_D E\left(Z^0, u\right) p\left(u\right) du$$

Statistical mean value

$$\left\langle E\left(Z^0, t\right)\right\rangle = \sum_{i=1}^{n} E_i\left(Z^0\right) w_i$$

$$w_i = \int_D L_i\left(u\right) p\left(u\right) du$$

Expected value of the output

$$\sigma_{\text{var}}^2 = \sum_{i=0}^{n} w_i E_i^2\left(Z^0\right) - \left(\sum_{i=0}^{n} w_i E_i^2\left(Z^0\right)\right)^2$$

Variance

Multiple independent random variables principle

Three random variables (height above ground, permittivity and field observation point) – projection on Lagrangian basis

$$I\left(X_1^0, X_2^0, X_3^0; r, s, t\right) \approx \sum_{i=0}^{n_1}\sum_{i=0}^{n_2}\sum_{i=0}^{n_3} I_{ijk}\left(X_1^0, X_2^0, X_3^0\right) L_i(r) L_j(s) L_k(t)$$

$$Influence(E)_i = \frac{Variance\left(E \middle| RV_i\right)}{Variance\left(E \middle| tot\right)}$$

Influence of each RV

Computational examples

Parameters of the calculation

- Height h

$$h = h^0 + \hat{u}_1^0 \qquad h^0 = X_1^0 = 0.3\,\text{m}$$

- zero-mean RV with uniform distribution from 0.05 to 0.55 m
- Ground permittivity ε_r

$$\varepsilon_r = \varepsilon_r^0 + \hat{u}_2^0 \qquad \varepsilon_r^0 = X_2^0 = 15$$

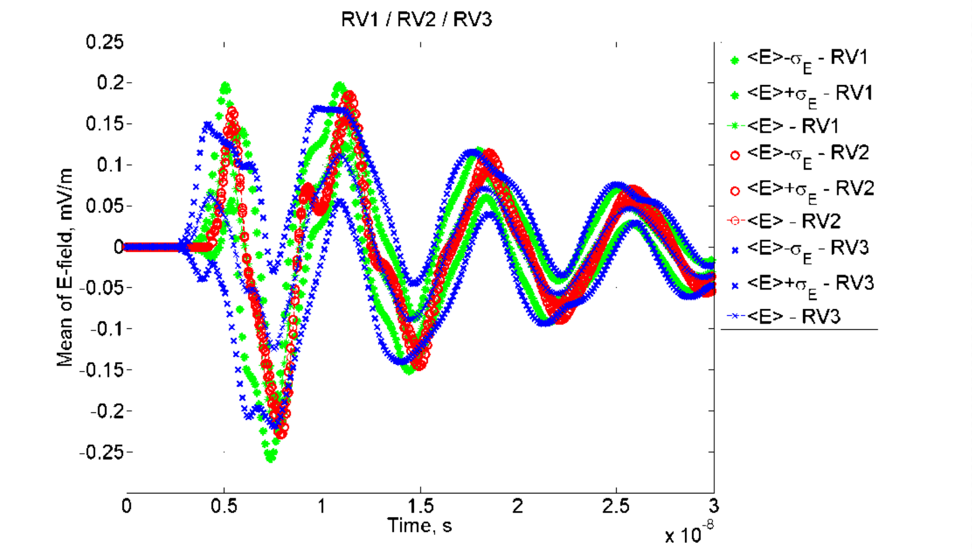
- zero-mean RV with uniform distribution from 2 to 28
- Observation point d

$$d = d^0 + \hat{u}_3^0 \qquad d^0 = X_3^0 = 1\,\text{m}$$

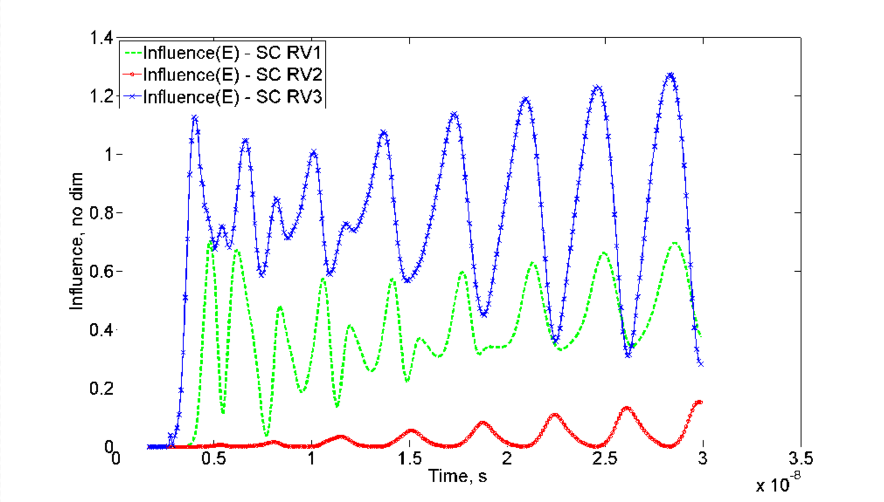
- zero-mean RV with uniform distribution from 0.5 to 1.5 m

Transmitted electric field – stochastic analysis

Mean value



Influence of each RV



Concluding remarks

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- Software SuzanaTD, freely available
- Analytical solutions for the wire antenna in the TD
  - Future work – solution for the transmitting antenna
- Transmitted electrical filed in the TD

Thank you for your attention!